

Variational inequalities for the fractional Laplacian

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Abstract

In this paper we study the obstacle problems for the fractional Laplacian of order $s \in (0, 1)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, under mild assumptions on the data.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 1$. Given $s \in (0, 1)$, a measurable function ψ and a distribution f on Ω , we consider the problem

$$\begin{cases} u \geq \psi & \text{in } \Omega \\ (-\Delta)^s u \geq f & \text{in } \Omega \\ (-\Delta)^s u = f & \text{in } \{u > \psi\} \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \quad (1.1)$$

Our interest is motivated by the noticeable paper [19], where Louis E. Silvestre investigated (1.1) in case $\Omega = \mathbb{R}^n$, $f = 0$ and ψ smooth. His results apply also to Dirichlet's problems on balls, see [19, Section 1.3]. Besides remarkable results, in [19] the interested reader can find stimulating motivations for (1.1), arising from

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mathematical finance. In addition, Signorini's problem, also known as the lower dimensional obstacle problem for the classical Laplacian, can be recovered from (1.1) by taking $s = \frac{1}{2}$.

Among the papers dealing with (1.1) and related problems we cite also [1, 3, 4, 7, 15, 18] and references there-in, with no attempt to provide a complete reference list.

In the present paper we show that the free boundary problem (1.1) admits a solution under quite mild assumptions on the data, see Theorems 1.1 and 1.2 below. However, our starting interest included broader questions concerning the variational inequality

$$u \in K_\psi^s, \quad \langle (-\Delta)^s u - f, v - u \rangle \geq 0 \quad \forall v \in K_\psi^s, \quad (\mathcal{P}(\psi, f))$$

where $f \in \tilde{H}^s(\Omega)'$ and

$$K_\psi^s = \left\{ v \in \tilde{H}^s(\Omega) \mid v \geq \psi \text{ a.e. on } \Omega \right\}.$$

Notation and main definitions are listed at the end of this introduction. We will always assume that the closed and convex set K_ψ^s is not empty, also when not explicitly stated.

Problem $\mathcal{P}(\psi, f)$ admits a unique solution u , that can be characterized as the unique minimizer for

$$\inf_{v \in K_\psi^s} \frac{1}{2} \langle (-\Delta)^s v, v \rangle - \langle f, v \rangle. \quad (1.2)$$

The variational inequality $\mathcal{P}(\psi, f)$ and the free boundary problem (1.1) are naturally related. Any solution $u \in \tilde{H}^s(\Omega)$ to (1.1) coincides with the unique solution to $\mathcal{P}(\psi, f)$, see Remark 3.5. Conversely, if u solves $\mathcal{P}(\psi, f)$ then $(-\Delta)^s u - f$ is a nonnegative distribution on Ω , compare with Theorem 3.2. By analogy with the local case $s = 1$ one can guess that $(-\Delta)^s u = f$ outside the coincidence set $\{u = \psi\}$, at least when u is regular enough. This is essentially the content of Section 3 in [19], where $f = 0$ and ψ is a smooth, rapidly decreasing function on $\Omega = \mathbb{R}^n$, and of Theorems 1.1, 1.2 below.

To study the variational inequality $\mathcal{P}(\psi, f)$ we took inspiration from the classical theory about the local case $s = 1$. In particular, we refer to the fundamental monograph [9] by Kinderlehrer and Stampacchia, and to the pioneering papers [2, 10, 11, 12, 13, 20, 21], among others.

Standard techniques do not apply directly in the fractional case, mostly because of the different behavior of the truncation operator $v \mapsto v^+$, $H^s(\mathbb{R}^n) \rightarrow H^s(\mathbb{R}^n)$. Section 2 is entirely devoted to this subject; we collect there some lemmata that might have an independent interest.

We take advantage of the results in Section 2 to obtain equivalent and useful formulations for $\mathcal{P}(\psi, f)$, and to prove continuous dependence theorems upon the data f and ψ , see Sections 3 and 4, respectively.

Some extra difficulties arise from having settled a nonlocal problem on a bounded domain, producing at least, but not only, the same (partially solved) technical difficulties as for the unconstrained problem $(-\Delta)^s u = f$, $u \in \tilde{H}^s(\Omega)$ (see for instance [6], [16], [17] and references there-in, for regularity issues).

Our main results proved in Section 5. They involve the unique solution ω_f to

$$(-\Delta)^s \omega_f = f \quad \text{in } \Omega, \quad \omega_f \in \tilde{H}^s(\Omega). \quad (1.3)$$

Theorem 1.1 *Assume that ψ and $f \in \tilde{H}^s(\Omega)'$ satisfy the following conditions:*

- A1) $(\psi - \omega_f)^+ \in \tilde{H}^s(\Omega)$;
- A2) $(-\Delta)^s(\psi - \omega_f)^+ - f$ is a locally finite signed measure on Ω ;
- A3) $((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \in L_{\text{loc}}^p(\Omega)$ for some $p \in [1, \infty]$.

Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then the following facts hold.

- i) $(-\Delta)^s u - f \in L_{\text{loc}}^p(\Omega)$;
- ii) $0 \leq (-\Delta)^s u - f \leq ((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \quad \text{a.e. on } \Omega$;
- iii) $(-\Delta)^s u = f \quad \text{a.e. on } \{u > \psi\}$.

In particular, u solves the free boundary problem (1.1).

Theorem 1.2 *Assume that Ω is a bounded Lipschitz domain satisfying the exterior ball condition. Let $\psi \in C^0(\overline{\Omega})$ be a given obstacle, such that K_ψ^s is not empty, $\psi \leq 0$ on $\partial\Omega$ and $f \in L^p(\Omega)$, for some exponent $p > n/2s$.*

Then the unique solution u to $\mathcal{P}(\psi, f)$ is continuous on \mathbb{R}^n and solves the free boundary problem (1.1).

Our results plainly cover the non-homogeneous Dirichlet's free boundary problem

$$\begin{cases} u \geq \psi & \text{in } \Omega \\ (-\Delta)^s u \geq f & \text{in } \Omega \\ (-\Delta)^s u = f & \text{in } \{u > \psi\} \\ u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

under appropriate assumptions on the datum g . Notice indeed that u solves the related variational inequality if and only if $u - g$ solves $\mathcal{P}(\psi - g, f + (-\Delta)^s g)$.

Free boundary problems for the operator $(-\Delta)^s u + u$ can be considered as well, with minor modifications in the statements and in the proofs.

Notation The definition of the fractional Laplacian $(-\Delta)^s$ involves the Fourier transform:

$$\mathcal{F}[(-\Delta)^s u] = |\xi|^{2s} \mathcal{F}[u], \quad \mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We adopt the standard notation

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in L^2(\mathbb{R}^n) \mid (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n)\}, \\ \tilde{H}^s(\Omega) &= \{u \in H^s(\mathbb{R}^n) \mid u \equiv 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega}\}. \end{aligned}$$

We endow $H^s(\mathbb{R}^n)$ and $\tilde{H}^s(\Omega)$ with their natural Hilbertian structures. We recall that the norm of u in $\tilde{H}^s(\Omega)$ is given by the $L^2(\mathbb{R}^n)$ -norm of $(-\Delta)^{\frac{s}{2}} u$.

We do not make any assumption on Ω . Thus $\partial\Omega$ might be very irregular, even a fractal, and $C_0^\infty(\Omega)$ might be not dense in $\tilde{H}^s(\Omega)$. Notice that $\tilde{H}^s(\Omega)$ coincides with $\tilde{H}^s(\Omega')$, whenever $\overline{\Omega} = \overline{\Omega}'$.

We denote by $\langle \cdot, \cdot \rangle$ the duality product between $\tilde{H}^s(\Omega)$ and its dual $\tilde{H}^s(\Omega)'$. In particular, $(-\Delta)^s u \in \tilde{H}^s(\Omega)'$ for any $u \in \tilde{H}^s(\Omega)$, and

$$\langle (-\Delta)^s u, v \rangle = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v dx = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}[u] \overline{\mathcal{F}[v]} d\xi.$$

2 Truncations

For measurable functions v, w we put, as usual,

$$v \vee w = \max\{v, w\}, \quad v \wedge w = \min\{v, w\}, \quad v^+ = v \vee 0, \quad v_- = -(v \wedge 0),$$

so that $v = v^+ - v_-$. It is well known that $v \vee w \in H^s(\mathbb{R}^n)$ and $v \wedge w \in H^s(\mathbb{R}^n)$ if $v, w \in H^s(\mathbb{R}^n)$.

Lemma 2.1 *Let $v \in H^s(\mathbb{R}^n)$. Then*

- i) $\langle (-\Delta)^s v^+, v^- \rangle = \langle (-\Delta)^s v^-, v^+ \rangle \leq 0;$
- ii) $\langle (-\Delta)^s v, v^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^-|^2 dx \leq 0;$
- iii) $\langle (-\Delta)^s v, v^+ \rangle - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^+|^2 dx \geq 0.$

In addition, if $v \in H^s(\mathbb{R}^n)$ does not have constant sign, then all the above inequalities are strict.

Proof. In [14, Theorem 6], the Caffarelli-Silvestre extension argument [5] has been used to check that

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} |v||^2 dx < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 dx,$$

whenever v changes sign. That is,

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ + v^-)|^2 dx < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ - v^-)|^2 dx.$$

The conclusion is immediate. □

Remark 2.2 *One can use ii) in Lemma 2.1 to get the well known weak maximum principle, that is, if $u \in \tilde{H}^s(\Omega)$ and $(-\Delta)^s u \geq 0$ in Ω then $u \geq 0$ in Ω .*

Corollary 2.3 *Let v_h be a sequence in $H^s(\mathbb{R}^n)$ such that v_h converges to a nonpositive function in $H^s(\mathbb{R}^n)$. Then $v_h^+ \rightarrow 0$ in $H^s(\mathbb{R}^n)$.*

Proof. Statement iii) in Lemma 2.1 provides the estimate

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v_h^+|^2 dx \leq \langle (-\Delta)^s v_h, v_h^+ \rangle, \quad (2.1)$$

that gives us the boundedness of the sequence v_h^+ in $H^s(\mathbb{R}^n)$. Since $v_h^+ \rightarrow 0$ in $L^2(\mathbb{R}^n)$, we have $v_h^+ \rightarrow 0$ weakly in $H^s(\mathbb{R}^n)$. Thus $\langle (-\Delta)^s v_h, v_h^+ \rangle \rightarrow 0$, as $(-\Delta)^s v_h$ converges in $H^s(\mathbb{R}^n)'$, and the conclusion follows from (2.1). □

Lemma 2.4 *Let $v \in \tilde{H}^s(\Omega)$ and $m > 0$. Then $(v + m)^-, (v - m)^+, v \wedge m \in \tilde{H}^s(\Omega)$ and*

$$\begin{aligned} i) \quad & \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(v + m)^-|^2 dx \leq 0; \\ ii) \quad & \langle (-\Delta)^s v, (v - m)^+ \rangle - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(v - m)^+|^2 dx \geq 0; \\ iii) \quad & \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(v \wedge m)|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}v|^2 dx - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(v - m)^+|^2 dx. \end{aligned}$$

Proof. Clearly, $(v + m)^- \in L^2(\mathbb{R}^n)$ and $(v + m)^- \equiv 0$ outside Ω . Fix a cutoff function $\eta \in C_0^\infty(\mathbb{R}^n)$, with $0 \leq \eta \leq 1$, and such that $\eta \equiv 1$ in a ball containing $\overline{\Omega}$. Then $(v + m)^- = (v + m\eta)^- \in \tilde{H}^s(\Omega)$, as trivially $m\eta \in H^s(\mathbb{R}^n)$.

For any integer $h \geq 1$ we set

$$\eta_h(x) = \eta\left(\frac{x}{h}\right),$$

so that $\eta_h \rightarrow 1$ pointwise. A direct computation shows that

$$(-\Delta)^s \eta_h(x) = h^{-2s} \left((-\Delta)^s \eta \right) \left(\frac{x}{h} \right) \rightarrow 0 \quad \text{in } L_{\text{loc}}^2(\mathbb{R}^n). \quad (2.2)$$

By *ii)* in Lemma 2.1 we have that

$$\begin{aligned} 0 & \geq \langle (-\Delta)^s(v + m\eta_h), (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(v + m)^-|^2 dx \\ & = \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(v + m)^-|^2 dx + m \int_{\Omega} ((-\Delta)^s \eta_h)(v + m)^- dx \\ & = \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(v + m)^-|^2 dx + o(1), \end{aligned}$$

by (2.2) and since $(v + m)^-$ has compact support in Ω . Claim *i)* is proved. To check *ii)* notice that $(v - m)^+ = ((-v) + m)^-$ and then use *i)* with $(-v)$ instead of v .

It remains to prove *iii)*. Notice that $v \wedge m = v - (v - m)^+$. Hence $v \wedge m \in \tilde{H}^s(\Omega)$. Using *ii)* we get

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}(v \wedge m)\|^2 & = \|(-\Delta)^{\frac{s}{2}}v\|^2 - 2\langle (-\Delta)^s v, (v - m)^+ \rangle + \|(-\Delta)^{\frac{s}{2}}(v - m)^+\|^2 \\ & \leq \|(-\Delta)^{\frac{s}{2}}v\|^2 - \|(-\Delta)^{\frac{s}{2}}(v - m)^+\|^2. \end{aligned}$$

The proof is complete. \square

3 Equivalent formulations

We start this section by introducing a crucial notion.

Definition 3.1 *A function $\mathcal{U} \in \tilde{H}^s(\Omega)$ is a supersolution for $(-\Delta)^s v = f$ if*

$$\langle (-\Delta)^s \mathcal{U} - f, \varphi \rangle \geq 0 \quad \text{for any } \varphi \in \tilde{H}^s(\Omega), \varphi \geq 0.$$

The above definition extends the usually adopted one in the local case $s = 1$, see [9, Definition 6.3]. A different definition of supersolution is used in [19] for $f = 0$. We refer to [19, Subsection 2.10], for a stimulating discussion on this subject.

Theorem 3.2 *Let $u \in K_\psi^s$. The following sentences are equivalent.*

- a) *u is the solution to problem $\mathcal{P}(\psi, f)$;*
- b) *u is the smallest supersolution for $(-\Delta)^s v = f$ in the convex set K_ψ^s . That is, $\mathcal{U} \geq u$ almost everywhere in Ω , for any supersolution $\mathcal{U} \in K_\psi^s$;*
- c) *u is a supersolution for $(-\Delta)^s v = f$ and*

$$\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0 \quad \text{for any } v \in K_\psi^s.$$

- d) *$\langle (-\Delta)^s v - f, v - u \rangle \geq 0$ for any $v \in K_\psi^s$.*

Proof. *a) \iff b). Assume that u solves $\mathcal{P}(\psi, f)$. Fix any nonnegative $\varphi \in \tilde{H}^s(\Omega)$. Testing $\mathcal{P}(\psi, f)$ with $u + \varphi \in K_\psi^s$ one gets $\langle (-\Delta)^s u - f, \varphi \rangle \geq 0$, that proves that u is a supersolution.*

Next, take any supersolution $\mathcal{U} \in K_\psi^s$. Then $u - (u - \mathcal{U})^+ = \mathcal{U} \wedge u \in K_\psi^s$. Thus

$$\langle (-\Delta)^s u - f, -(u - \mathcal{U})^+ \rangle \geq 0.$$

On the other hand, from $(-\Delta)^s \mathcal{U} - f \geq 0$ we get

$$\langle (-\Delta)^s \mathcal{U} - f, (u - \mathcal{U})^+ \rangle \geq 0.$$

Adding the above inequalities we arrive at

$$0 \geq \langle (-\Delta)^s (u - \mathcal{U}), (u - \mathcal{U})^+ \rangle \geq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (u - \mathcal{U})^+|^2 dx,$$

thanks to *iii*) in Lemma 2.1. Thus $(u - \mathcal{U})^+ = 0$ almost everywhere in Ω , that is, $u \leq \mathcal{U}$ and proves that *a*) implies *b*).

Conversely, assume that u satisfies *b*) and let \tilde{u} be the solution to $\mathcal{P}(\psi, f)$. We already know that $a) \Rightarrow b)$. Thus u and \tilde{u} must coincide, because both obey the condition of being the smallest supersolution to $(-\Delta)^s v = f$ in K_ψ^s . Hence, *a*) holds.

a) \iff c). Let u be the solution to $\mathcal{P}(\psi, f)$. We already know that u is supersolution. Fix any function $v \in K_\psi^s$. Notice that

$$u + (v - u)^- \geq u \geq \psi, \quad u - (v - u)^- = v \wedge u \geq \psi.$$

Thus, testing $\mathcal{P}(\psi, f)$ with $u \pm (v - u)^-$ we get $\langle (-\Delta)^s u - f, \pm (v - u)^- \rangle \geq 0$, that is, *c*) holds.

Conversely, assume that u satisfies *c*). Let $\tilde{u} \in K_\psi^s$ be the solution to $\mathcal{P}(\psi, f)$. We already proved that \tilde{u} is the smallest supersolution in K_ψ^s . In particular, $\tilde{u} \leq u$ and thus

$$\langle (-\Delta)^s u - f, u - \tilde{u} \rangle = \langle (-\Delta)^s u - f, (\tilde{u} - u)^- \rangle = 0$$

by the assumption *c*) on u . Since \tilde{u} solves $\mathcal{P}(\psi, f)$, we also get

$$\langle (-\Delta)^s \tilde{u} - f, u - \tilde{u} \rangle \geq 0.$$

Subtracting, we infer $\langle (-\Delta)^s (u - \tilde{u}), u - \tilde{u} \rangle \leq 0$, that is, $u = \tilde{u}$.

a) \iff d). Clearly *a*) implies *d*) because

$$\begin{aligned} & \langle (-\Delta)^s v - f, v - u \rangle \\ &= \langle (-\Delta)^s u - f, v - u \rangle + \langle (-\Delta)^s (v - u), v - u \rangle \geq \langle (-\Delta)^s u - f, v - u \rangle. \end{aligned}$$

Now assume that u satisfies *d*) and fix any $v \in K_\psi^s$. From $\frac{v+u}{2} \in K_\psi^s$ and *d*) we obtain

$$\begin{aligned} 0 &\leq 2 \langle (-\Delta)^s \left(\frac{v+u}{2} \right) - f, \frac{v+u}{2} - u \rangle = \frac{1}{2} \langle (-\Delta)^s (v+u), v-u \rangle - \langle f, v-u \rangle \\ &= \left(\frac{1}{2} \langle (-\Delta)^s v, v \rangle - \langle f, v \rangle \right) - \left(\frac{1}{2} \langle (-\Delta)^s u, u \rangle - \langle f, u \rangle \right). \end{aligned}$$

Thus u solves the minimization problem (1.2), that is, u solves $\mathcal{P}(\psi, f)$. \square

Remark 3.3 In the local case $s = 1$, the equivalence between a) and d) is commonly known as Minty's lemma, see [13].

Corollary 3.4 Let $f_1, f_2 \in \tilde{H}^s(\Omega)'$ and let u_i be the solution to $\mathcal{P}(\psi, f_i)$, $i = 1, 2$. If $f_1 \geq f_2$ in the sense of distributions, then $u_1 \geq u_2$ a.e. in Ω .

Proof. The function u_1 is a supersolution for $(-\Delta)^s v = f_2$ and $u_1 \in K_\psi^s$. Hence $u_1 \geq u_2$, by statement b) in Theorem 3.2. \square

Remark 3.5 Let $u \in \tilde{H}^s(\Omega)$ be a solution to (1.1). Then $(-\Delta)^s u - f$ can be identified with a nonnegative Radon measure on Ω having support in $\{u = \psi\}$. If $v \in K_\psi^s$, then $(v - u)^-$ vanishes on $\{u = \psi\}$. Thus $\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0$, hence u solves $\mathcal{P}(\psi, f)$ by Theorem 3.2.

4 Continuous dependence results

Theorem 4.1 Let ψ_1, ψ_2 be given obstacles, $f \in \tilde{H}^s(\Omega)'$ and let u_i be the solution to $\mathcal{P}(\psi_i, f)$, $i = 1, 2$. If $\psi_1 - \psi_2 \in L^\infty(\Omega)$, then $u_1 - u_2$ is bounded, and

$$i) \quad \|(u_1 - u_2)^+\|_\infty \leq \|(\psi_1 - \psi_2)^+\|_\infty, \quad ii) \quad \|(u_1 - u_2)^-\|_\infty \leq \|(\psi_1 - \psi_2)^-\|_\infty.$$

Proof. Put $m := \|(\psi_1 - \psi_2)^+\|_\infty$. Since $(u_2 - u_1 + m)^- \in \tilde{H}^s(\Omega)$ by Lemma 2.4, then

$$v_1 := u_1 - (u_2 - u_1 + m)^- = (u_2 + m) \wedge u_1 \in K_{\psi_1}^s.$$

Hence we can use v_1 as test function in $\mathcal{P}(\psi_1, f)$ to get

$$\langle (-\Delta)^s u_1 - f, -(u_2 - u_1 + m)^- \rangle \geq 0.$$

On the other hand, we can test $\mathcal{P}(\psi_2, f)$ with $u_2 + (u_2 - u_1 + m)^- \in K_{\psi_2}^s$. Hence

$$\langle (-\Delta)^s u_2 - f, (u_2 - u_1 + m)^- \rangle \geq 0.$$

Adding and taking i) of Lemma 2.4 into account, we arrive at

$$-\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(u_2 - u_1 + m)^-|^2 dx \geq \langle (-\Delta)^s(u_2 - u_1), (u_2 - u_1 + m)^- \rangle \geq 0.$$

Hence, $(u_2 - u_1 + m)^- = 0$. We have proved that $(u_1 - u_2)^+ \leq m$ a.e. in Ω , hence i) holds. Inequality ii) can be proved in the same way. \square

Corollary 4.2 *Let $\psi \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$, with $p \in (1, \infty)$, $p > n/2s$. Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then $u \in L^\infty(\Omega)$ and*

$$\psi \vee \omega_f \leq u \leq \|\psi^+\|_\infty + c\|f^+\|_p \quad \text{a.e. in } \Omega, \quad (4.1)$$

where ω_f solves (1.3) and c depends only on n, s, p and Ω . In particular, if $f = 0$ then

$$\psi^+ \leq u \leq \|\psi^+\|_\infty.$$

Proof. First of all, notice that $f \in \tilde{H}^s(\Omega)'$ by Sobolev embedding theorem. Since u is supersolution of (1.3), the first inequality in (4.1) follows by the maximum principle in Remark 2.2.

Denote by ω_{f^+} the unique solution to (1.3) with f replaced by f^+ . If $n > 2s$ we use convolution to define

$$U = c_1|x|^{2s-n} * (f^+ \cdot \chi_\Omega).$$

For proper choice of the constant c_1 , U solves $(-\Delta)^s U = f^+ \cdot \chi_\Omega$ in \mathbb{R}^n . Convolution estimates give $U \leq c\|f^+\|_p$ on \mathbb{R}^n . By the maximum principle, $\omega_{f^+} \leq U$ on Ω , hence $\omega_{f^+} \leq c\|f^+\|_p$. For $n = 1 \leq 2s$ this inequality also holds, see, e.g., [16, Remark 1.5].

Now let u_1 be the unique solution of $\mathcal{P}(\psi, f^+)$. Then $u_1 \geq u$ by Corollary 3.4. Finally, we can consider ω_{f^+} as the solution of the problem $\mathcal{P}(\omega_{f^+}, f^+)$. Theorem 4.1 gives

$$u \leq (u_1 - \omega_{f^+})^+ + \omega_{f^+} \leq \|(\psi - \omega_{f^+})^+\|_\infty + \omega_{f^+},$$

and the last inequality in (4.1) follows. \square

Roughly speaking, Theorem 4.1 concerns the continuity of $L^\infty \ni \psi \mapsto u \in L^\infty$. The next result gives the continuity of the arrow $L^\infty \ni \psi \mapsto u \in \tilde{H}^s(\Omega)$.

Theorem 4.3 *Let $\psi_h \in L^\infty(\Omega)$ be a sequence of obstacles and let $f \in \tilde{H}^s(\Omega)'$ be given. Assume that there exists $v_0 \in \tilde{H}^s(\Omega)$, such that $v_0 \geq \psi_h$ for any h .*

Denote by u_h the solution to the obstacle problem $\mathcal{P}(\psi_h, f)$. If $\psi_h \rightarrow \psi$ in $L^\infty(\Omega)$, then $u_h \rightarrow u$ in $\tilde{H}^s(\Omega)$, where u is the solution to the limiting problem $\mathcal{P}(\psi, f)$.

Proof. Let u be the solution to $\mathcal{P}(\psi, f)$. We already know from Theorem 4.1 that $\|u - u_h\|_\infty \leq \|\psi - \psi_h\|_\infty$. Hence, in particular, $u_h \rightarrow u$ a.e. in Ω . Now, test $\mathcal{P}(\psi_h, f)$

with v_0 to obtain that

$$\langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h - f, v_0 \rangle + \langle f, u_h \rangle.$$

Hence, the sequence u_h is bounded in $\tilde{H}^s(\Omega)$. Therefore, $u_h \rightarrow u$ weakly in $\tilde{H}^s(\Omega)$. To prove that $u_h \rightarrow u$ in the $\tilde{H}^s(\Omega)$ norm we only need to show that

$$\limsup_{h \rightarrow \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2 \leq \| (-\Delta)^{\frac{s}{2}} u \|_2.$$

For any $\varepsilon > 0$ we introduce the function

$$v_\varepsilon = u + (v_0 - u) \wedge \varepsilon.$$

Since $\psi_h \rightarrow \psi$ in $L^\infty(\Omega)$, we have $v_\varepsilon \geq \psi_h$ for h large enough. Using v_ε as test function in $\mathcal{P}(\psi_h, f)$ we get

$$\langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon - u_h \rangle \geq 0,$$

and hence

$$\| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 = \langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u_h \rangle.$$

Letting $h \rightarrow \infty$ we infer

$$\begin{aligned} \limsup_{h \rightarrow \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 &\leq \langle (-\Delta)^s u - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u \rangle \\ &= \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + \langle (-\Delta)^s u - f, (v_0 - u) \wedge \varepsilon \rangle. \end{aligned} \quad (4.2)$$

Now we let $\varepsilon \rightarrow 0$. Clearly $(v_0 - u) \wedge \varepsilon \rightarrow -(v_0 - u)^-$ in $L^2(\Omega)$. In addition, the functions $(v_0 - u) \wedge \varepsilon$ are uniformly bounded in $\tilde{H}^s(\Omega)$ by *iii* in Lemma 2.4. Thus $(v_0 - u) \wedge \varepsilon \rightarrow -(v_0 - u)^-$ weakly in $\tilde{H}^s(\Omega)$. Thus, from (4.2) we get

$$\limsup_{h \rightarrow \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 \leq \| (-\Delta)^{\frac{s}{2}} u \|_2^2 - \langle (-\Delta)^s u - f, (v_0 - u)^- \rangle = \| (-\Delta)^{\frac{s}{2}} u \|_2^2$$

since u solves $\mathcal{P}(\psi, f)$, and therefore it satisfies condition *c*) in Theorem 3.2. Thus $u_h \rightarrow u$ in $\tilde{H}^s(\Omega)$. \square

Next we deal with the continuity of the arrow $H^s \ni \psi \mapsto u \in \tilde{H}^s$.

Theorem 4.4 *Let $\psi_h \in H^s(\mathbb{R}^n)$ be a sequence of obstacles such that $\psi_h^+ \in \tilde{H}^s(\Omega)$, and let f_h be a sequence in $\tilde{H}^s(\Omega)'$. Assume that*

$$\psi_h \rightarrow \psi \quad \text{in } H^s(\mathbb{R}^n), \text{ and } f_h \rightarrow f \quad \text{in } H^s(\Omega)'.$$

Denote by u_h the solution to the obstacle problem $\mathcal{P}(\psi_h, f_h)$. Then $u_h \rightarrow u$ in $\tilde{H}^s(\Omega)$, where u is the solution to the limiting obstacle problem $\mathcal{P}(\psi, f)$.

Proof. We can assume that $f_h, f = 0$. If not, replace the obstacles ψ_h and ψ with $\psi_h - \omega_{f_h}$ and $\psi - \omega_f$, respectively, see (1.3).

Let u_h solve $\mathcal{P}(\psi_h, 0)$ and let u be the solution to the limiting problem $\mathcal{P}(\psi, 0)$. Recall that u is the unique minimizer for

$$\inf_{v \in K_\psi^s} \langle (-\Delta)^s v, v \rangle. \quad (4.3)$$

Since $u \vee \psi_h = u + (\psi_h - u)^+$ and $\psi_h - u \rightarrow \psi - u \leq 0$, then

$$u \vee \psi_h \rightarrow u \quad \text{in } \tilde{H}^s(\Omega) \quad (4.4)$$

by Corollary 2.3. Moreover, $u \vee \psi_h \in K_{\psi_h}^s$ and thus from $\mathcal{P}(\psi_h, 0)$ we infer

$$\langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h, u \vee \psi_h \rangle. \quad (4.5)$$

Inequality (4.5) guarantees the boundedness of the sequence u_h in $\tilde{H}^s(\Omega)$. Hence we can assume that $u_h \rightarrow \tilde{u}$ weakly in $\tilde{H}^s(\Omega)$. Since $\psi_h \rightarrow \psi$ and $u_h \rightarrow \tilde{u}$ a.e. in Ω , clearly $\tilde{u} \in K_\psi^s$.

Next, by weak lower semicontinuity, (4.5) and (4.4) we get

$$\langle (-\Delta)^s \tilde{u}, \tilde{u} \rangle \leq \liminf_{h \rightarrow \infty} \langle (-\Delta)^s u_h, u_h \rangle \leq \limsup_{h \rightarrow \infty} \langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s \tilde{u}, u \rangle. \quad (4.6)$$

Thus

$$\|(-\Delta)^{\frac{s}{2}} \tilde{u}\|_2^2 \leq \|(-\Delta)^{\frac{s}{2}} \tilde{u}\|_2 \|(-\Delta)^{\frac{s}{2}} u\|_2.$$

Hence, $\tilde{u} = u$, as the minimization problem (4.3) admits a unique solution, and (4.6) implies $\|(-\Delta)^{\frac{s}{2}} u_h\|_2 \rightarrow \|(-\Delta)^{\frac{s}{2}} u\|_2$. Hence $u_h \rightarrow u$ strongly in $\tilde{H}^s(\Omega)$. \square

5 Proof of the main results

We start with a preliminary theorem of independent interest, that gives distributional bounds on $(-\Delta)^s u - f$ under mild assumptions on the data.

Theorem 5.1 *Let ψ and $f \in \tilde{H}^s(\Omega)'$ satisfying assumptions A1) and A2) in Theorem 1.1. Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then*

$$0 \leq (-\Delta)^s u - f \leq ((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \quad \text{in the distributional sense on } \Omega.$$

Proof. The main tool was inspired by the penalty method by Lewy-Stampacchia [10] and already used for instance in [18] under smoothness assumptions on the data and on the solution.

In order to simplify notations we start the proof with some remarks. First, we can assume that $f = 0$, as we did in the proof of Theorem 4.4. Thus $(-\Delta)^s u \geq 0$ and $u \geq \psi$, that imply $u \geq \psi^+$, use the maximum principle in Remark 2.2. Clearly u is the smallest supersolution to $(-\Delta)^s v = 0$ in $K_{\psi^+}^s$, and hence it solves the obstacle problem $\mathcal{P}(\psi^+, 0)$. In conclusion, it suffices to prove Theorem 5.1 in case $f = 0$ and $\psi \geq 0$ in \mathbb{R}^n . Our aim is to show that

$$0 \leq (-\Delta)^s u \leq ((-\Delta)^s \psi)^+ \quad \text{in the distributional sense on } \Omega, \quad (5.1)$$

for $\psi \in \tilde{H}^s(\Omega)$, $\psi \geq 0$, such that $(-\Delta)^s \psi$ is a measure on Ω .

The proof of (5.1) will be achieved in few steps.

Step 1 *Assume $(-\Delta)^s \psi \in L^p(\Omega)$ for any large exponent $p > 1$. Then (5.1) holds.*

We take $p \geq \frac{2n}{n+2s}$, that is needed only if $n > 2s$. Then $\tilde{H}^s(\Omega) \hookrightarrow L^{p'}(\Omega)$ and $L^p(\Omega) \subset \tilde{H}^s(\Omega)'$ by Sobolev embeddings. In particular $((-\Delta)^s \psi)^+ \in \tilde{H}^s(\Omega)'$.

Take a function $\theta_\varepsilon \in C^\infty(\mathbb{R})$ such that $0 \leq \theta_\varepsilon \leq 1$, and

$$\theta_\varepsilon(t) = 1 \quad \text{for } t \leq 0, \quad \theta_\varepsilon(t) = 0 \quad \text{for } t \geq \varepsilon.$$

By standard variational methods we have that there exists a unique $u_\varepsilon \in \tilde{H}^s(\Omega)$ that weakly solves

$$(-\Delta)^s u_\varepsilon = \theta_\varepsilon(u_\varepsilon - \psi) ((-\Delta)^s \psi)^+ \quad \text{in } \Omega.$$

We claim that

$$u \leq u_\varepsilon \leq u + \varepsilon \quad \text{a.e. in } \Omega.$$

By *iii)* in Lemma 2.1 we can estimate

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}(\psi - u_\varepsilon)^+\|_2^2 &\leq \langle (-\Delta)^s(\psi - u_\varepsilon), (\psi - u_\varepsilon)^+ \rangle \\ &\leq \int_{\Omega} ((-\Delta)^s \psi)^+ (1 - \theta_\varepsilon(u_\varepsilon - \psi)) (\psi - u_\varepsilon)^+ dx = 0. \end{aligned}$$

Hence, $u_\varepsilon \geq \psi$. Since $(-\Delta)^s u_\varepsilon \geq 0$, then $u_\varepsilon \geq u$ by *b)* in Theorem 3.2. Next, we use *iii)* in Lemma 2.4 and $(-\Delta)^s u \geq 0$ to estimate

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}(u_\varepsilon - u - \varepsilon)^+\|_2^2 &\leq \langle (-\Delta)^s(u_\varepsilon - u), (u_\varepsilon - u - \varepsilon)^+ \rangle \\ &\leq \int_{\Omega} ((-\Delta)^s \psi)^+ \theta_\varepsilon(u_\varepsilon - \psi) (u_\varepsilon - u - \varepsilon)^+ dx = 0. \end{aligned}$$

Thus $u_\varepsilon \leq u + \varepsilon$, and the claim is proved. In particular, we have that $\|u_\varepsilon - u\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, for any nonnegative test function $\varphi \in C_0^\infty(\Omega)$ we have that

$$\begin{aligned} \langle (-\Delta)^s u, \varphi \rangle &= \int_{\Omega} u (-\Delta)^s \varphi dx = \int_{\Omega} u_\varepsilon (-\Delta)^s \varphi dx + o(1) \\ &= \langle (-\Delta)^s u_\varepsilon, \varphi \rangle + o(1) \leq \langle ((-\Delta)^s \psi)^+, \varphi \rangle + o(1), \end{aligned}$$

that readily gives $(-\Delta)^s u \leq ((-\Delta)^s \psi)^+$ in the distributional sense in Ω .

Step 2 *Approximation argument.*

Fix a small $\varepsilon > 0$ and put $\Omega_\varepsilon := \{x \in \Omega \mid \text{dist}(x, \Omega) < \varepsilon\}$. The convex set

$$K_\varepsilon = \{v \in \tilde{H}^s(\Omega_\varepsilon) \mid v \geq \psi \text{ a.e. on } \mathbb{R}^n\}$$

contains K_ψ^s , hence it is not empty. We denote by u_ε the unique solution to the variational inequality

$$u_\varepsilon \in K_\varepsilon, \quad \langle (-\Delta)^s u_\varepsilon, v - u_\varepsilon \rangle \geq 0 \quad \forall v \in K_\varepsilon, \quad (\mathcal{P}_\varepsilon)$$

so that $u_\varepsilon \in \tilde{H}^s(\Omega_\varepsilon)$ and is nonnegative. Next we prove that

$$0 \leq (-\Delta)^s u_\varepsilon \leq ((-\Delta)^s \psi)^+ \quad \text{in the distributional sense on } \Omega. \quad (5.2)$$

For, we approximate ψ in a standard way, via convolution. Let $(\rho_h)_h$ be a sequence of mollifiers such that $\text{supp}(\rho_h) \subset B_{\frac{1}{h}}$ and put $\psi_h = \psi * \rho_h$. Notice that for h large enough, $\psi_h = 0$ outside Ω_ε . Therefore

$$\psi_h \in \tilde{H}^s(\Omega_\varepsilon), \quad \psi_h \rightarrow \psi \quad \text{in } H^s(\mathbb{R}^n). \quad (5.3)$$

The convex set $K_{\varepsilon,h} := \{v \in \tilde{H}^s(\Omega_\varepsilon) \mid v \geq \psi_h\}$ is not empty, as it contains ψ_h . The variational inequality

$$u_h \in K_{\varepsilon,h}, \quad \langle (-\Delta)^s u_h, v - u_h \rangle \geq 0 \quad \forall v \in K_{\varepsilon,h}, \quad (\mathcal{P}_{\varepsilon,h})$$

has a unique solution $u_h \in \tilde{H}^s(\Omega_\varepsilon)$. Theorem 4.4 readily gives that $u_h \rightarrow u_\varepsilon$ in $\tilde{H}^s(\Omega_\varepsilon)$. Since $(-\Delta)^s \psi_h \in L^p(\mathbb{R}^n)$ for any $p \geq 1$, then Step 1 applies. In particular

$$0 \leq (-\Delta)^s u_h \leq ((-\Delta)^s \psi_h)^+ \quad \text{in the distributional sense on } \Omega. \quad (5.4)$$

Next, $((-\Delta)^s \psi)^+ * \rho_h$ is a nonnegative smooth function, and

$$((-\Delta)^s \psi)^+ * \rho_h \geq ((-\Delta)^s \psi) * \rho_h = (-\Delta)^s \psi_h.$$

Thus $((-\Delta)^s \psi)^+ * \rho_h \geq ((-\Delta)^s \psi_h)^+$, and (5.4) implies

$$0 \leq (-\Delta)^s u_h \leq ((-\Delta)^s \psi)^+ * \rho_h \quad \text{in the distributional sense on } \Omega.$$

Claim (5.2) follows, since $((-\Delta)^s \psi)^+ * \rho_h \rightarrow ((-\Delta)^s \psi)^+$ in the sense of measures, and $(-\Delta)^s u_h \rightarrow (-\Delta)^s u_\varepsilon$ in the sense of distributions.

Step 3 Conclusion of the proof.

The last step in the proof consists in passing to the limit along a sequence $\varepsilon \rightarrow 0$. First, we notice that $u \in \tilde{H}^s(\Omega_\varepsilon)$ and in particular $u \in K_\varepsilon$. Therefore, using the variational characterization of the unique solution u_ε to $(\mathcal{P}_\varepsilon)$ we find

$$\frac{1}{2} \langle (-\Delta)^s u_\varepsilon, u_\varepsilon \rangle \leq \frac{1}{2} \langle (-\Delta)^s u, u \rangle. \quad (5.5)$$

Now we fix $\varepsilon_0 > 0$. Thanks to (5.5), we get that the sequence u_ε is bounded in $\tilde{H}^s(\Omega_{\varepsilon_0})$, and therefore we can assume that $u_\varepsilon \rightarrow \tilde{u}$ weakly in $\tilde{H}^s(\Omega_{\varepsilon_0})$. From (5.5) we readily get

$$\frac{1}{2} \langle (-\Delta)^s \tilde{u}, \tilde{u} \rangle \leq \frac{1}{2} \langle (-\Delta)^s u, u \rangle. \quad (5.6)$$

On the other hand, $u_\varepsilon \rightarrow \tilde{u}$ almost everywhere. Hence $\tilde{u} \in \tilde{H}^s(\Omega)$ and $\tilde{u} \geq \psi$ on Ω , that is, $\tilde{u} \in K_\psi^s$. Using the characterization of u as the unique solution to the minimization problem (4.3), from (5.6), (5.5) we get that $\tilde{u} = u$ and $u_\varepsilon \rightarrow u$ in $\tilde{H}^s(\Omega_{\varepsilon_0})$. In particular, $\langle (-\Delta)^s u_\varepsilon, \varphi \rangle \rightarrow \langle (-\Delta)^s u, \varphi \rangle$ for any $\varphi \in C_0^\infty(\Omega)$. Now, from (5.2) we know that $((-\Delta)^s \psi)^+ - (-\Delta)^s u_\varepsilon$ is a nonnegative distribution on Ω . Thus $((-\Delta)^s \psi)^+ - (-\Delta)^s u$ is nonnegative as well, and (5.1) is proved. \square

Proof of Theorem 1.1

Statements *i*) and *ii*) hold by Theorem 5.1. It remains to prove the last claim.

It is not restrictive to assume $f \equiv 0$. Hence u solves $\mathcal{P}(\psi, 0)$, $(-\Delta)^s u \geq 0$ by Theorem 3.2, and u is nonnegative in Ω , see Remark 2.2. Actually u is lower semicontinuous and positive by the strong maximum principle, see for instance [8, Theorem 2.5]. Thus $u \geq \psi^+$ and $\{u > \psi\} = \{u > \psi^+\}$.

Next we use *c*) in Theorem 3.2 with $v = \psi^+ \in \tilde{H}^s(\Omega)$, to get

$$\langle (-\Delta)^s u, u - \psi^+ \rangle = 0.$$

Let Ω' be any domain compactly contained in Ω . We claim that

$$\int_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+) dx = 0. \quad (5.7)$$

Since $(-\Delta)^s u \cdot (u - \psi^+)$ is a measurable nonnegative function then the integral in (5.7) is nonnegative. To prove the opposite inequality we put $g_m = (u - \psi^+) \wedge m$, $m \geq 1$. Let φ be any nonnegative cut off function, with $\varphi \in C_0^\infty(\Omega)$ and $\varphi \equiv 1$ on Ω' . Since $(-\Delta)^s u \geq 0$, $(-\Delta)^s u \in L_{\text{loc}}^1(\Omega)$, $u - \psi^+ \geq \varphi g_m$ and $\varphi g_m \in L^\infty(\Omega)$ has compact support in Ω , we have that

$$0 = \langle (-\Delta)^s u, u - \psi^+ \rangle \geq \langle (-\Delta)^s u, \varphi g_m \rangle = \int_{\Omega} (-\Delta)^s u \cdot (\varphi g_m) dx \geq \int_{\Omega'} (-\Delta)^s u \cdot g_m dx.$$

Next, use the monotone convergence theorem to get

$$0 \geq \lim_{m \rightarrow \infty} \int_{\Omega'} (-\Delta)^s u \cdot g_m dx = \int_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+) dx,$$

that concludes the proof of (5.7).

Now, since Ω' was arbitrarily chosen and $(-\Delta)^s u \cdot (u - \psi^+) \geq 0$, equality (5.7) implies that $(-\Delta)^s u \cdot (u - \psi^+) = 0$ a.e. in Ω , and *iii*) is proved. \square

Remark 5.2 *Theorem 1.1 holds with the same proof also in the local case $s = 1$. Notice that no regularity assumptions on Ω are needed, and the cases $p = 1, p = \infty$ are included as well.*

Remark 5.3 *To obtain better regularity results for u , one can apply the regularity theory for*

$$(-\Delta)^s u = g \in L^p(\Omega) \quad \text{in } \Omega, \quad u \in \tilde{H}^s(\Omega).$$

In particular, if $p > \frac{n}{2s}$ and Ω is Lipschitz and satisfies the exterior ball condition, then u is Hölder continuous in Ω . See for example [16, Proposition 1.4] and [17, Proposition 1.1].

Proof of Theorem 1.2

As usual, we can assume $f = 0$. Fix a small $\varepsilon > 0$, and let ψ_h^ε be a mollification of $\psi - \varepsilon$. Then ψ_h^ε is smooth on $\overline{\Omega}$, $\psi_h^\varepsilon < 0$ on $\partial\Omega$ and $\psi_h^\varepsilon \rightarrow \psi - \varepsilon$ uniformly on $\overline{\Omega}$, as $h \rightarrow \infty$.

By Theorem 1.1, the solution $u_h \in \tilde{H}^s(\Omega)$ to $\mathcal{P}(\psi_h^\varepsilon, 0)$ satisfies $(-\Delta)^s u_h^\varepsilon \in L^p(\Omega)$ and therefore u_h^ε is Hölder continuous, see Remark 5.3. Moreover, the estimates in Theorem 4.1 imply that $u_h^\varepsilon \rightarrow u^\varepsilon$ uniformly on Ω , where u^ε solves $\mathcal{P}(\psi - \varepsilon, 0)$. In particular, $u^\varepsilon \in C^0(\overline{\Omega})$. Finally, use again Theorem 4.1 to get that $u^\varepsilon \rightarrow u$ uniformly, where u solves $\mathcal{P}(\psi, 0)$. In particular, u is continuous on \mathbb{R}^n .

To check the last statement notice that the set $\{u > \psi\} \subseteq \Omega$ is open; for any test function $\varphi \in C^\infty(\{u > \psi\})$ we have that $u \pm t\varphi \in K_\psi^s$ and therefore $t\langle (-\Delta)^s u, \pm\varphi \rangle \geq 0$ for $|t|$ small enough. The conclusion is immediate. \square

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